

Connecting N -representability to Weyl's problem: The one particle density matrix for $N = 3$ and $R = 6$

Mary Beth Ruskai *

Department of Mathematics, Tufts University, Medford, MA 02155

Marybeth.Ruskai@tufts.edu

February 1, 2008

Abstract

An analytic proof is given of the necessity of the Borland-Dennis conditions for 3-representability of a one particle density matrix with rank 6. This may shed some light on Klyachko's recent use of Schubert calculus to find general conditions for N -representability.

1 Introduction

The recent announcement by A. Klyachko [8] of the solution of the pure state N -representability problem for fermionic one-particle density matrix observes that this is the first new result since the work of Borland and Dennis [2] in the early 1970's. There may therefore be some historical value in unpublished work of the author from that time, which makes a connection between the Borland-Dennis conditions and Weyl's problem. The latter asks for conditions on sequences $\{a_k\}, \{b_k\}, \{c_k\}$ which ensure that there exist self-adjoint matrices A, B, C with eigenvalues a_k, b_k, c_k respectively such that $A + B = C$. The first complete solution to Weyl's problem was given by Klyachko [7] in 1998.

Let γ be a density matrix normalized so that $\text{Tr } \gamma = N$. The pure state N -representability problem for fermions asks for necessary and sufficient conditions on γ for the existence of an antisymmetric N -particle state whose one-particle reduced density matrix is γ . Let R denote the rank of γ . For the case $N = 3$ and $R = 6$, Borland and Dennis gave a pair of conditions on the eigenvalues λ_k of γ which can

*Partially supported by the National Science Foundation under Grant DMS-0604900

be written as follows under the assumption that they are arranged in non-increasing order.

$$\lambda_1 + \lambda_6 = 1, \quad \lambda_2 + \lambda_5 = 1, \quad \lambda_3 + \lambda_4 = 1 \quad (1)$$

$$\lambda_1 + \lambda_2 \leq \lambda_3 + 1 \quad (2)$$

Note that (1) can be written compactly as $\lambda_k + \lambda_{7-k} = 1$ for $k = 1, 2, 3$

Borland and Dennis [2] proposed their conditions on the basis of numerical studies and gave a proof of (2) under an assumption, which is equivalent to (1), about the pre-image of γ . In this note, we show that (1) is a necessary condition for N -representability when $N = 3$ and $R = 6$, completing the analytic proof of Borland and Dennis. We begin with some background material in Section 2. In Section 3 we present a proof of the necessity of (1). In Section 4 we give a different, independent proof of the necessity of the inequality (2) from Weyl's inequalities. For completeness, we include a proof of sufficiency of (1) and (2) in Section 5. In Sections 6 and 7 we present some partial results for the cases $N = 3$ and $R = N + 3$ in the hope of providing some intuition behind the success of Klyachko's approach to a full solution.

2 Notation and background

In this note, we write the eigenvectors of γ as $|\phi_k\rangle$ so that

$$\gamma = \sum_k \lambda_k |\phi_k\rangle \langle \phi_k|. \quad (3)$$

We will let \mathcal{A} denote the anti-symmetrization operator and use the notation $[f_j, f_k, f_\ell] = \mathcal{A}f_j(x_1)f_k(x_2)f_\ell(x_3)$ to denote a Slater determinant. The notation $\langle \cdot, \cdot \rangle_m$ indicates a partial inner product on a tensor product of Hilbert spaces.

We need some results from Section 10 of Coleman's fundamental paper [3]. The first is Theorem 10.6 in [3]

Lemma 1. (Coleman) *The one-particle density matrix γ is N -representable with pre-image $|\Psi\rangle = \sqrt{\lambda_1} \mathcal{A}|\phi_1\rangle \otimes |\Phi_1\rangle + \sqrt{1-\lambda_1} |\Phi_2\rangle$ if and only if it can be written in the form*

$$\gamma = \lambda_1 |\phi_1\rangle \langle \phi_1| + \lambda_1 \gamma_1 + (1 - \lambda_1) \gamma_2 \quad (4)$$

where γ_1 is the $(N-1)$ -representable reduced density matrix of $|\Phi_1\rangle$, and γ_2 is N -representable with pre-image Φ_2 satisfying

$$\langle \phi_1, \Phi_2 \rangle_1 = \langle \Phi_1, \Phi_2 \rangle_{2,3\dots N} = 0. \quad (5)$$

The next two results are Theorems 10.2 and 10.4 respectively in [3]. (See also [10].)

Theorem 2. *A one-particle density matrix γ is 2-representable if and only if all non-zero eigenvalues are doubly degenerate. If there are no other degeneracies and the eigenvalues are written in non-increasing order so that $\lambda_{2k-1} = \lambda_{2k} > \lambda_{2k+1}$, then the pre-image of γ must have the form*

$$|\Psi\rangle = \sum_k e^{i\theta_k} \sqrt{\lambda_{2k}} [\phi_{2k-1}, \phi_{2k}] \quad (6)$$

Theorem 3. *When $N = 2n + 1$ is odd and the one-particle density matrix γ has rank $R = N + 2$, it is N -representable if and only if $\lambda_1 = 1$ and the remaining eigenvalues are doubly degenerate.*

3 Necessity of the condition $\lambda_k + \lambda_{7-k} = 1$.

To show that (1) is a necessary condition for 3-representability when $R = 6$, observe that since $\lambda_1 = \langle \phi_1, \gamma \phi_1 \rangle$ it follows from (4) that

$$\langle \phi_1, \gamma_1 \phi_1 \rangle = \langle \phi_1, \gamma_2 \phi_1 \rangle = 0.$$

Therefore, γ_1 and γ_2 have rank ≤ 5 . It then follows from Theorem 3 that one can write

$$\gamma_2 = |g_1\rangle\langle g_1| + |a|^2|g_2\rangle\langle g_2| + |a|^2|g_3\rangle\langle g_3| + |b|^2|g_4\rangle\langle g_4| + |b|^2|g_5\rangle\langle g_5|$$

with $|a|^2 + |b|^2 = 1$ and $|\Phi_2\rangle = a[g_1, g_2, g_3] + b[g_1, g_4, g_5]$. There is no loss of generality in writing $\Phi_1 = \sum_{j < k} c_{jk}[g_j, g_k]$.

We first consider the case in which both $a, b \neq 0$. Then a simple computation shows that (5) implies

$$|\Phi_1\rangle = c_{24}[g_2, g_4] + c_{25}[g_2, g_5] + c_{34}[g_3, g_4] + c_{35}[g_3, g_5]$$

so that $\langle g_1, \Phi_1 \rangle_1 = 0$. Defining $|\phi_6\rangle = |g_1\rangle$, gives $\lambda_6 = 1 - \lambda_1$ and one can rewrite (4) as

$$\gamma = \lambda_1 |\phi_1\rangle\langle \phi_1| + (1 - \lambda_1) |\phi_6\rangle\langle \phi_6| + \lambda_1 \gamma_1 + (1 - \lambda_1) \tilde{\gamma}_2 \quad (7)$$

where $\tilde{\gamma}_2 = \gamma_2 - |g_1\rangle\langle g_1|$ is the reduced density matrix of $|G_1\rangle = \langle g_1, \Phi_2 \rangle_3 = a[g_2, g_3] + b[g_4, g_5]$. Thus, in the orthonormal basis $\{g_2, g_3, g_4, g_5\}$ we find

$$\gamma_1 = \begin{pmatrix} |c_{24}|^2 + |c_{25}|^2 & \bar{c}_{24}c_{34} + \bar{c}_{25}c_{35} & 0 & 0 \\ c_{24}\bar{c}_{34} + c_{25}\bar{c}_{35} & |c_{34}|^2 + |c_{35}|^2 & 0 & 0 \\ 0 & 0 & |c_{24}|^2 + |c_{34}|^2 & \bar{c}_{24}c_{25} + \bar{c}_{34}c_{35} \\ 0 & 0 & c_{24}\bar{c}_{25} + c_{34}\bar{c}_{35} & |c_{25}|^2 + |c_{35}|^2 \end{pmatrix}.$$

The key point is that γ_1 is block diagonal and can be diagonalized by a block diagonal unitary transformation which mixes only within pairs g_2, g_3 and g_4, g_5 leaving the Slater determinants in G_1 unaffected (except possibly for a phase factor which can be absorbed into the new basis). Denoting the new basis as ϕ_k , we now have $|G_1\rangle = a[\phi_2, \phi_3] + b[\phi_4, \phi_5]$. Then either by explicit computation or from Coleman's proof [4] of Theorem 2, one can write $|\Phi_1\rangle = s[\phi_2, \phi_4] + t[\phi_3, \phi_5]$ with $|s|^2 + |t|^2 = 1$. Thus, the eigenvalues of γ satisfy

$$\lambda_2 = \lambda_1|a|^2 + (1 - \lambda_1)|s|^2 \quad (8a)$$

$$\lambda_3 = \lambda_1|b|^2 + (1 - \lambda_1)|s|^2 \quad (8b)$$

$$\lambda_4 = \lambda_1|a|^2 + (1 - \lambda_1)|t|^2 \quad (8c)$$

$$\lambda_5 = \lambda_1|b|^2 + (1 - \lambda_1)|t|^2 \quad (8d)$$

which implies

$$\lambda_2 + \lambda_5 = \lambda_3 + \lambda_4 = \lambda_1 + (1 - \lambda_1) = 1. \quad (9)$$

We now consider the possibility that one of a, b is zero, in which case, $|\Phi_2\rangle$ is a single Slater determinant and there is no loss of generality in writing as $\Phi_2 = [g_1, g_2, g_3]$. Then (5) implies that one can write

$$|\Phi_1\rangle = \sum_{j=1,2,3} \sum_{k=4,5} x_{jk} [g_j, g_k] + c[g_4, g_5] \quad (10)$$

Now regard x_{jk} as a 3×2 matrix and observe when U, V are 3×3 and 2×2 unitary matrices, $Y = UXV^\dagger$ corresponds to a basis change which mixes g_1, g_2, g_3 and g_4, g_5 among themselves. By the singular value decomposition we can find U, V such that only y_{24} and y_{35} are non-zero. Thus, in the new basis which we call ϕ_k

$$|\Phi_1\rangle = y_{24}[\phi_2, \phi_4] + y_{35}[\phi_3, \phi_5] + c[\phi_4, \phi_5]. \quad (11)$$

Again writing $\phi_6 = g_1$, we find that the pre-image of γ has the form

$$|\Psi\rangle = a_{123}[\phi_1, \phi_2, \phi_3] + a_{246}[\phi_2, \phi_4, \phi_6] + a_{356}[\phi_3, \phi_5, \phi_6] + a_{456}[\phi_4, \phi_5, \phi_6] \quad (12)$$

which implies (1).

4 Necessity of the inequality (2)

We now prove that the inequality (2) is necessary for N -representability. When γ has the form (3) and (1) holds, its pre-image can be written in the form

$$\begin{aligned} |\Psi\rangle = & x_{000}[\phi_1, \phi_2, \phi_3] + x_{001}[\phi_1, \phi_2, \phi_4] + x_{010}[\phi_1, \phi_5, \phi_3] + x_{011}[\phi_1, \phi_5, \phi_4] \\ & + x_{100}[\phi_6, \phi_2, \phi_3] + x_{101}[\phi_6, \phi_2, \phi_4] + x_{110}[\phi_6, \phi_5, \phi_3] + x_{111}[\phi_6, \phi_5, \phi_4]. \end{aligned} \quad (13)$$

In this form, there is no loss of generality in assuming that the λ_k are arranged in non-increasing order. If we now define

$$S = \begin{pmatrix} x_{000} & x_{001} \\ x_{010} & x_{011} \end{pmatrix} \quad T = \begin{pmatrix} x_{100} & x_{101} \\ x_{110} & x_{111} \end{pmatrix} \quad (14)$$

then the reduced density matrix of $|\Psi\rangle$ is (up to a permutation) $W_1 \oplus W_2 \oplus W_3$ with

$$W_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_6 \end{pmatrix} = \begin{pmatrix} \text{Tr } SS^\dagger & \text{Tr } ST^\dagger \\ \text{Tr } TS^\dagger & \text{Tr } TT^\dagger \end{pmatrix} \quad (15)$$

$$W_2 = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_5 \end{pmatrix} = SS^\dagger + TT^\dagger \quad (16)$$

$$W_3 = \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{pmatrix} = S^\dagger S + T^\dagger T \quad (17)$$

It follows from (15) that the eigenvalues of SS^\dagger , which are the same as those of $S^\dagger S$ can be written as $\sigma, \lambda_1 - \sigma$ with $0 \leq \sigma \leq \lambda_1$; similarly those of TT^\dagger and $T^\dagger T$ can be written as $\tau, \lambda_6 - \tau$ with $0 \leq \tau \leq \lambda_6$.

The form of (16) and (17) is suggestive of Weyl's problem with $A = SS^\dagger, B = TT^\dagger, C = W_2$ in the case of (16) and adjoints reversed for (17). Weyl [6, 11] used the max-min principle to find necessary conditions

$$a_1 + b_1 \geq c_1, \quad a_2 + b_1 \geq c_2, \quad a_1 + b_2 \geq c_2 \quad (18)$$

(with all three sequences in non-increasing order). For 2×2 matrices satisfying $\text{Tr } A + \text{Tr } B = \text{Tr } C$, these are also sufficient. We apply Weyl's inequalities to (16) and (17) and retain the stronger in each pair to obtain

$$\sigma + \tau \geq \lambda_2 \quad (19a)$$

$$\lambda_1 - \sigma + \tau \geq \lambda_4 \quad (19b)$$

$$\sigma + \lambda_6 - \tau \geq \lambda_4 \quad (19c)$$

Adding together the first two inequalities implies

$$2\tau \geq \lambda_2 + \lambda_4 - \lambda_1. \quad (20)$$

Combining this with $2\lambda_6 \geq 2t$ and using (1) gives

$$2(1 - \lambda_1) = 2\lambda_6 \geq \lambda_2 + 1 - \lambda_3 - \lambda_1 \quad (21)$$

which is equivalent to (2).

5 Sufficiency

To prove sufficiency, it suffices to consider a pre-image of the form

$$\Psi = \hat{a}[\phi_1, \phi_2, \phi_3] + \hat{b}[\phi_1, \phi_4, \phi_5] + \hat{s}[\phi_6, \phi_2, \phi_4] + \hat{t}[\phi_6, \phi_5, \phi_3] \quad (22)$$

and observe that its first order reduced density matrix is diagonal in the basis ϕ_k with

$$|\hat{a}|^2 + |\hat{b}|^2 = \lambda_1 \quad |\hat{s}|^2 + |\hat{t}|^2 = \lambda_6.$$

Under the assumption that (1) holds, the linear relation between the eigenvalues of γ and $|\hat{a}|^2, |\hat{b}|^2, |\hat{s}|^2, |\hat{t}|^2$ can be inverted to yield

$$|\hat{a}|^2 = \frac{1}{2}(\lambda_2 + \lambda_3 - \lambda_6) \quad |\hat{b}|^2 = \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_4) \quad (23a)$$

$$|\hat{s}|^2 = \frac{1}{2}(\lambda_2 - \lambda_3 + \lambda_6) \quad |\hat{t}|^2 = \frac{1}{2}(\lambda_6 - \lambda_2 + \lambda_3). \quad (23b)$$

With the ordering convention $\lambda_k \geq \lambda_{k+1}$, the expressions for $|\hat{a}|^2, |\hat{b}|^2$ and $|\hat{s}|^2$ are all positive; and $|\hat{t}|^2 \geq 0$ is equivalent to (2).

In Section 3, we showed slightly more than that (1) holds. We also showed that the pre-image can always be written in a form in which only four of the coefficients in (13) are non-zero. However, neither of these forms is equivalent to (22) with λ_k decreasing. The equations for the coefficients in one of those forms could have solutions only when a stronger inequality than (2) holds. In particular, the form obtained from (7) in the paragraph before (8) has solutions only when $\lambda_1 + \lambda_2 \leq \lambda_4 + 1$.

6 General $R = N + 3$ with N odd

It is tempting to try to extend the argument in Section 3 to the general case of $R = N + 3$ when N is odd. Using (4) we can conclude as before that γ_2 must be N -representable with $R = N + 2$ and thus has an eigenvector $|g_1\rangle$ with eigenvalue 1. We can write its pre-image as

$$\begin{aligned} |\Phi_2\rangle &= a_m[g_1, g_2, g_3, \dots, g_{N-1}, g_N] + \dots + a_k[g_1, g_2, g_3, \dots, g_{2k-1}, g_{2k+2}, \dots, g_{N-1}, g_N] \\ &\quad + \dots + a_1[g_1, g_4, g_4, \dots, g_{N+1}, g_{N+2}] \end{aligned} \quad (24)$$

where $m = \frac{1}{2}(N + 1)$ and a_k is the coefficient of the Slater determinant which does *not* contain g_{2k} or g_{2k+1} . However, it is not evident that the strong orthogonality condition $\langle g_1, \Phi_1 \rangle_1 = 0$ holds as was the case for $N = 3$. If we knew that

$$\lambda_1 + \langle g_1, \gamma g_1 \rangle \leq 1, \quad (25)$$

strong orthogonality would follow, and we could again conclude that g_1 is an eigenvector of γ with eigenvalue $\langle g_1, \gamma g_1 \rangle = 1 - \lambda_1$.

For the case $N = 5, R = N + 3$ with N odd, Altunbulak and Klyachko [1] have shown that $\lambda_1 + \lambda_R \leq 1$. This is not equivalent to (25) because we don't know that g_1 is the eigenfunction for λ_R . A condition of the form $\lambda_j + \lambda_{j'} \leq 1$ is sometimes called a “strong Pauli condition”. When the largest eigenvalue is non-degenerate, we can show that strong orthogonality implies a strong Pauli condition with equality. This suggests the following

Conjecture 4. *When N is odd and $R = N + 3$, a necessary condition for pure state N -representability of a one-particle density matrix is $\lambda_1 + \lambda_R = 1$, where we have assumed that the eigenvalues are in non-increasing order.*

Proposition 5. *Let $R = N + 3$ with N odd and consider the decomposition (4) of a one-particle density matrix γ under the assumption that λ_1 is the largest eigenvalue. Then $|\Phi_2\rangle$ has an eigenvector $|g_1\rangle$ with eigenvalue 1. If $\langle g_1, \Phi_1 \rangle_1 = 0$, then $|g_1\rangle$ is an eigenvector of γ with eigenvalue $1 - \lambda_1$ and this is the smallest eigenvalue of γ .*

Proof: Let $|\phi_k\rangle$ denote an eigenvector of γ orthogonal to both $|\phi_1\rangle$ and $|g_1\rangle$, and write

$$\begin{aligned} |\Phi_1\rangle &= a \mathcal{A}|\phi_k \otimes \chi_1\rangle + \sqrt{1 - a^2} |\psi_1\rangle \\ |\Phi_2\rangle &= b \mathcal{A}|g_1 \otimes \phi_k \otimes \chi_2\rangle + \sqrt{1 - b^2} |g_1 \otimes \psi_2\rangle. \end{aligned}$$

where we have absorbed any phases into ψ_j . Then $\lambda_k = \lambda_1 a^2 + (1 - \lambda_1) b^2$. Since each $|\psi_j\rangle$ is strongly orthogonal to $|\phi_1\rangle, |g_1\rangle$ and $|\phi_k\rangle$, each $|\psi_j\rangle$ is an $(N - 1)$ -particle function with one-rank at most N . It is well-known [3, 5, 10] that this implies that $|\psi_j\rangle$ is a single Slater determinant. Since both $|\psi_1\rangle$ and $|\psi_2\rangle$ have one-particle density matrices in the same N -dimensional subspace, it follows that the ranges of these one-particle density matrices have a non-zero intersection. Let $|f\rangle$ be in this intersection. Then

$$\langle f, \gamma f \rangle = (1 - a^2) \lambda_1 + (1 - b^2) (1 - \lambda_1) = 1 - \lambda_k \quad (26)$$

Thus, if, $\lambda_k < 1 - \lambda_1$, then $\langle f, \gamma f \rangle > \lambda_1$ contradicting the assumption that λ_1 is the largest eigenvalue. **QED**

7 Further connections with Weyl's problem

Now assume that g_1 is strongly orthogonal to Φ_1 and, as in (7), write

$$\gamma = \lambda_1 |\phi_1\rangle \langle \phi_1| + (1 - \lambda_1) |g_1\rangle \langle g_1| + \lambda_1 \gamma_1 + (1 - \lambda_1) \tilde{\gamma}_2. \quad (27)$$

The N -representability problem in this situation is reduced to finding conditions which ensure that a density matrix is a convex combination of two $(N - 1)$ -representable

density matrices of rank $N + 1$ which satisfy an additional orthogonality constraint. Write

$$\begin{aligned} |\Phi_1\rangle &= \sum_{k_1 < k_2 < \dots < k_{N-1}} x_{k_1 k_2 \dots k_{N-1}} [g_1, g_2, \dots, g_{N-1}] \\ |\Phi_2\rangle &= \sum_{k_1 < k_2 < \dots < k_{N-1}} y_{k_1 k_2 \dots k_{N-1}} [g_1, g_2, \dots, g_{N-1}] \end{aligned}$$

Let X, Y be the corresponding anti-symmetric tensors, and let

$$XY^\dagger = \sum_{k_2, k_3, \dots, k_M} x_{k_1, k_2, \dots, k_M} \bar{y}_{k_1, k_2, \dots, k_M} \quad (28)$$

denote contraction over $k_2 \dots k_M$. Then, we can rewrite (7) as

$$\gamma - \lambda_1 |\phi_1\rangle\langle\phi_1| - (1 - \lambda_1) |g_1\rangle\langle g_1| = XX^\dagger + YY^\dagger \quad (29)$$

with the constraint $\langle\Phi_1, \Phi_2\rangle = \text{Tr } XY^\dagger = 0$. This is a constrained version of Weyl's problem. If the $R = N + 3$ problem could be solved in this way, then by particle-hole duality, we would also have the solution to the 3-representability problem. Although we do not know if strong orthogonality of g_1 to Φ_1 holds in general, this viewpoint provides a connection to Weyl's problem that is more general the situation for which it was used in Section 4.

For general R (or for $R = N + 3$ without the simplification that leads to (7)), Coleman's Lemma 1 gives a constrained version of Weyl's problem with $\gamma_1 = XX^\dagger$ and $\gamma_2 = YY^\dagger$. But now γ_1 is $(N-1)$ -representable and γ_2 is N -representable and the orthogonality condition (5) must be translated to tensors of different size. Nevertheless, it now seems clear that what Coleman referred to as a double induction lemma, was a constrained version of Weyl's problem. The solution to Weyl's problem was given less than 10 years ago, with more recent refinements [9]. Thus, it is not surprising that the pure state N -representability problem also resisted solution and that Klyachko succeeded by using powerful techniques associated with Schubert calculus to solve both problems.

Acknowledgment: It is a pleasure to recall that most of this work was the result of discussions with R.E. Borland and K. Dennis during a visit to the National Physical Laboratory in Great Britain in the fall of 1970.

References

- [1] M. Altunbulak and A. Klyachko, private communication
- [2] R.E. Borland and K. Dennis, “The conditions on the one-matrix for three-body fermion wavefunctions with one-rank equal to six” *J. Phys. B* **5**, 7–15 (1972).
- [3] A.J. Coleman, “The structure of fermion density matrices” *Rev. Mod. Phys.* **35**, 668–687 (1963).
- [4] A.J. Coleman, “The structure of fermion density matrices” Uppsala report No. 80 (June, 1962).
- [5] L. L. Foldy, “Antisymmetric Functions and Slater Determinants” *J. Math. Phys.* **3**, 531–538 (1962)
- [6] R.A. Horn and C.R. Johnson, *Matrix Analysis* (Cambridge University press, 1985).
- [7] A. Klyachko, “Stable bundles, representation theory and Hermitian operators” *Selecta Math.* **4**, 419–445 (1998).
- [8] A. Klyachko, “Quantum marginal problem and N-representability” *Journal of Physics: Conf. Series* **36**, 72–86 (2006). quant-ph/0511102
- [9] A. Knutson and T. Tao, “Honeycombs and sums of Hermitian matrices” *Notices of AMS* **48**, 175–186 (2001).
- [10] M.B. Ruskai, “ N -representability problem: Particle-hole equivalence” *J. Math. Phys.* **11**, 3218–3224 (1970).
- [11] H. Weyl, “Das asymptotische verteilungsgesetz der eigenwerte linearer linearer partieller differentialgleichungen” *Math. Ann.* **71**, 441–479 (1912).